## Graph Theory and Combinatorics

Midterm Examination

February 22, 2025

**Instructions:** All questions carry ten marks. All graphs are assumed to be simple.

- 1. Let N be the incidence matrix of a symmetric  $2 (v, k, \lambda)$  design. Prove that its transpose  $N^t$  is also the incidence matrix of a 2-design.
- 2. Prove that a bipartite graph has a unique bipartition (upto interchanging the parts) if and only if it is connected.
- 3. Let  $V(G) = A \sqcup B$  be the partition of a bipartite graph G. Assume that |N(S)| > |S| for every non-empty subset S of A. Then prove that every edge of G belongs to some matching of of size |A|.
- 4. Let  $A_1, A_2, \ldots A_m$  be subsets of a set S such that for every  $J \subset \{1, 2, \ldots m\}$ , we have

$$|\cup_{j\in J} A_j| \ge |J|.$$

Then prove that we can choose  $x_j \in A_j$  such that  $x_i \neq x_l$  for all  $1 \leq i \neq l \leq m$ .

5. Let G be a graph and let S, T be subsets of its vertex set. Prove that the maximum number of vertx disjoint S, T-paths is equal to the minimum size of an S, T-separating set.